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ON WHETHER A T- STEINER QUINTUPLE SYSTEM OF BALANCED INCOMPLETE BLOCK DESIGN IS A GROUP, RING OR FIELD ALGEBRA

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ABSTRACT

This paper seeks to establish if the t- Steiner quintuple system of balanced incomplete block design (BIBD) is a group, ring or field algebra. A 2- (11, 5, 2) BIBD was constructed with its blocks, incidence matrix and Cayley table shown and the axioms of the algebraic structures of group, ring and field defined. The t- Steiner quintuple design represented as G_E was tested with the axioms of the algebraic structures. The results showed that a t – Steiner quintuple system of balanced incomplete block design satisfied all the axioms of a group under the additive binary operation but breaks down under the multiplicative binary operation. Results further showed that the t-Steiner quintuple design satisfied all the axioms of a ring algebra, a semi- group, commutative semi-ring and commutative ring but did not satisfied the axioms of a field algebra. This goes to show that a t- Steiner system is a group algebra under addition, a ring but neither a group under multiplication nor a field algebra.

Key words: Algebraic Structure, Balanced Designs, Binary Operation, Steiner Quintuple Designs

INTRODUCTION

Designs theory is the study of the existence, construction and examination of the properties of finite sets whose arrangement satisfies some concepts of balance and symmetry (Akra et al, 2024). A block design is a non-empty N = $\{n_1, n_2, n_3........n_m\}$ whose elements are varieties or elements and a non-empty collection of subsets of b blocks of size p with each of the elements replicated or appears r times in the blocks. According to Akra et al (2025), a block design is incomplete if the number of varieties is greater than the block size, that is $(p < n)$. An incomplete block design is said to be balanced when its consist of a set of points N that is divided into b subsets in such a way that each point in N is contained in r different subsets and the couple of points in N is contained in $(\lambda < b)$ subsets with $(p < n)$ points in each subset. A Balanced Incomplete Block Design (BIBD) is traditionally known by the parameters n, p, r, b, λ where n is the varieties or elements, p is the block size, r is the number each element is replicated in the blocks, b is the total number of blocks and λ is the pair of treatment (Akra et al, 2025). These parameters are connected by the formula $bp = nr$ and $\lambda(n-1) = r(p-1)$ known as the Fisher's formula (Fisher 1940). Another approach for construction of BIBDs is done by (Akra et al 2021).

On the other hand, algebraic structures such as group, ring and field algebra are mathematical systems that consist of a set of elements, S of cardinality $|S|$ and one or more operations $\langle \oplus, \otimes \rangle$ defined on that set. These structures follow specific rules or axioms depending on the type of structure.

Algebra structures come into play in the proof of the existence and construction of balanced incomplete block design or generally, block design because block designs make extensive use of discrete mathematics or algebra most especially Combinatorics. Combinatorial designs provides a solid foundation in the classical area of design theory as well as in many contemporary designs-based applications in a variety of fields and they are related to algebraic concepts of group, ring and field. Akra et al (2023) has worked on the algebraic structure for BIBD and came out with some useful findings.

As one of the foundational discrete structures, combinatorial design is one of the fastest growing area of modern mathematics and has wide range of applications especially in design analysis and model fittings (Michael et al 2025, 2017). Other areas of usefulness include cryptography and information security, mobile and wireless communication, DNA screening, software and hardware testing etc.

The aim of this paper is therefore to ascertain whether a t-Steiner quintuple system of balanced incomplete block design is a group, ring or a field algebra. The knowledge of it will help a great deal in scientific researches.

Steiner Systems of balanced incomplete block designs are special balanced incomplete block designs that is very much concerned with the number of elements in the block. It was first proposed and constructed by Sir Jacob Steiner (1853) and studied by Plucker (1835), Kirkman (1857), Cayley (1850), Bay and de Weck (1935), Bose (1939, 1942), Skolen (1958), Hannani (1961, 1975) and reported by the Encyclopedia of design theory (2004). Kirkman (1857) showed that the triple (3 elements in each block) system of order p exists and is balanced if and only if n ≡ 1, 3 (mod 6). Steiner (1853) studied the triple system, that is, S (n, 3, λ) and proposed the problem of arranging 'n' objects in triplets such that every pair of objects appears in precisely one triplet. That arrangement is a balanced incomplete block design and because his work was more broadly disseminated mathematical circles, the triple system was named after him.

The study of Steiner triple systems started with STS(n, p λ) for n = 7 and 9, p or block size or cardinality as 3 and λ = 1. By extending the work further, STS (n) of other orders such as 19, 21 etc. and λ increased from one to two were proved of their existences and subsequently constructed. Nevertheless, in all the extended and constructed designs, p was a constant 3.

Moore (1896) posed the problem of the existence of a Steiner quadruple systems, S (v, p, λ) in which p = 4. Barrau (1908) worked on it and established the uniqueness and existence of the p = 4 family of BIBDs. He equally constructed S (8, 4, 3) and S (10, 4, 3) which is a 6k + 2 and 6k + 4 for k = 1 respectively as was later shown by Hannani (1954). Fitting

(1915) made an in roads and constructed S (26, 4, 3) and S (34, 4, 3) using the cyclic approach. Bays and de Weck (1935) followed the work of Moore and proved the existence of at least one (14, 4, 3). The greatest breakthrough in the study of the Steiner quadruple systems came, when Hanani (1960) gave the necessary and sufficient conditions for the existence of such a system to be n $\equiv$ 2 or 4 (mod 6) which can be interpreted as n = 6k + 2 or 6k + 4.

A Steiner System, S (n, p, λ)- BIBDs in which p or cardinality or block size is 5 is called a Steiner quintuple system and there are quite a few literature in the quintuple systems of balanced incomplete design. The man behind the introduction of the p =5 Steiner systems is Hanani (1972) and because it follows the Steiner systems, it was called the Steiner quintuple systems written as S (n, 5,λ)-. He gave the necessary (not sufficient) condition for the existence of such a system as $n \equiv 5$ (mod 6) which comes from considerations that applies to all the classical Steiner systems. An additional necessary condition is that $n \neq$ 4 (mod 5), which comes from the fact that the number of blocks must be an integer. Hannai (1972) also proved that for balanced incomplete block designs with blocks having five elements each (p = 5), the known necessary existence condition is also sufficient, with the exception of the non-existing design S (15, 5, 2). Nevertheless, the sufficient condition for the quintuple family of Steiner systems is still under investigation. This is so because there is a quintuple of order 11, that is, n = 6k + 5 for k = 1 but there is no quintuple of order 17, that is, n = 6k + 5 for k = 2. There are quintuple for order 23, that is for k = 3, but no quintuple for order 29, that is, k = 4, there is a quintuple for order 35, that is, for k = 5, but there is no quintuple for order 41, that is , k = 6 etc. Unlike the triple and quadruple systems where at least λ and n have been varied, work on the quintuple only revolves around the order n.

It has been established that in a BIBD, every pair {x, y} appears together λ times. This pair is what is referred to as t. Though initially, this pair was equal to two in the early designs like in Kirkman (7,3,1) , Bose (9,3,1) , Steiner (7, 3,1) , Kirkman (7, 3, 2) and Denniston (15, 3, 2) designs, yet, since it was not regarded as an important parameter, it was not attached to other BIBD parameters. While the parameters of all BIBDs including the Steiner systems BIBDs are supposed to incorporate t so that BIBD would be written as t-(n, p, λ) -BIB designs, yet, that was never the case. In those early designs, the parameters of interest were, n, p, b, r, λ .

The study of t- designs is of recent. Authors like Earl Krasmer and Dale Menser (1976), Wilson (1973) , Ho Y. S and Mendelsohn (1974) and Hananni (1972) attempted to link the parameters of the BIBD with t and mentioned it passively but did not do an in-depth study on it. The most useful result on t- design was given by Peter Keevash (2014) when he proposed the natural divisibility conditions for a t- (n, p, λ)-BIBD and gave it to be $\binom{p-i}{t-i} / \lambda \binom{n-i}{t-i}$.

This is largely reported by Joy Moris (2021) in her book, Combinatorics where he posited that the divisibility theorem is a necessary condition for the existence of t- systems of designs. He further argued that the designs that do not obey the divisibility theorem does not exist. The relationship between t, b, p, r and λ was proposed by Keevash (2014) and is stated thus;

$\binom{p-i}{t-i}$ is a divisor of $\lambda\binom{n-i}{t-i}$ for every $0 \le i \le t-1$ (1)

Therefore a t-design or t –Steiner systems of balanced incomplete block design written as $t-(n, p, \lambda)$-BIB design, is a pair (X, B), where X is an n-set of elements and B is a system of p-sets (called blocks) from X such that each t -set is in exactly λ.

MATERIALS AND METHODS

Methodology

The first thing is the construction a t-Steiner quintuple design and in this work, our design is a 2-(11, 5, 2) BIBD.

$X_E = \{0,1,2,3,4,5,6,7,8,9,10\}$.

The initial block is (02348) and blocks are generated cyclically, thus, the 11 blocks are;

(13459) (245610) (35670) (46781) (57892) (689103) (079104) (081015) (90126) (101237)

And is put into block as; l

Table 1: Design Block

B$_1$	B$_2$	B$_3$	B$_4$	B$_5$	B$_6$	B$_7$	B$_8$	B$_9$	B$_{10}$	B$_{11}$
0	1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
8	9	10	0	1	2	3	4	5	6	7

Its incidence matrix, the design is thus constructed and shown;

$$\begin{bmatrix} 1&0&0&1&0&0&0&1&1&1&0 \\ 0&1&0&0&1&0&0&0&1&1&1 \\ 1&0&1&0&0&1&0&0&0&1&1 \\ 1&1&0&1&0&0&1&0&0&0&1 \\ 1&1&1&0&1&0&0&1&0&0&0 \\ 0&1&1&1&0&1&0&0&1&0&0 \\ 0&0&1&1&1&0&1&0&0&1&0 \\ 0&0&0&1&1&1&0&1&0&0&1 \\ 1&0&0&0&1&1&1&0&1&0&0 \\ 0&1&0&0&0&1&1&1&0&1&0 \\ 0&0&1&0&0&0&1&1&1&0&1 \end{bmatrix}$$

From the incidence matrix above, we see that $p(t_2,t_1), p(t_3,t_2), p(t_4,t_5), p(t_5,t_4).........p(t_{10},t_9), \ = \lambda = 2$, p = 5, r = 5, and n = 11 which satisfies both the conditions for a BIBD and all the axioms of a t – Steiner quintuple Design. Thus, we have successfully constructed a 2 - STS (11,5,2)

Definitions of Algebraic Structures of Group, Ring and Field

Definition and Axioms of a Group Algebra

According to Akra et al (2023), a group is a single system of $\langle G, \otimes \rangle$ or $\langle G, \oplus \rangle$ where G is a non–empty set and $\otimes$ or $\oplus$ is a multiplicative and additive binary operation on G. In other words, a group of finite number of elements is called a finite group. The order of a group, G, denoted by 0(G) and expressed mathematically as $|G|$, is the number of distinct elements in G. The followings are the axioms of a Group algebra;

G1: Closure: $\forall x,\ y \in G$, then $x \otimes y \in G$ and $\forall x,\ y \in G$, then $x \oplus y \in G$

G2: Associativity: $\forall x,\ y,\ z \in G,$; $x(y \otimes z) = (x \otimes y)z$ and $x(y \oplus z) = (x \oplus y)z$

G3: Existence of identity: There is an element, $e \in G,$ called an identity s.t $x \otimes e = x = e \otimes x\ \forall x \in G,$ and $x \oplus e = x = e \oplus x\ \forall x \in G,$

G4: Existence of inverse: For each $x \in G,$ there is an element $x^{-1} \in G,$ called an inverse of x s.t $x \otimes x^{-1} = e = x^{-1} \otimes x$ and $x \oplus x^{-1} = e = x^{-1} \oplus x$

G5: Commutative law: $\forall\ x,\ y \in G;\ \ x \otimes y = y \otimes x$ and $x,\ y \in G;\ \ x \oplus y = y \oplus x$ (abelian or commutative group).

i. A group $\langle G, \otimes \rangle$ or $\langle G, \oplus \rangle$ for which the postulate G5 does not hold is called a non – abelian group.
ii. If G is finite, then $\langle G, \otimes \rangle$ or $\langle G, \oplus \rangle$ is called a finite group, otherwise, it is called an infinite group.
iii. A system $\langle G, \otimes \rangle$ consisting of a non –empty set G and a binary composition $\otimes$ on G is called a semi-group if it satisfies the associativity axiom.

Definition and Axioms of a Ring Algebra

A Ring algebra is a double system, $\langle R, \oplus, \otimes \rangle$, where R is a non – empty set, and $\oplus$, $\oplus$ are two binary operations defined on the set R (Akra et al, 2023). A Ring have the following axioms;

R1: Closure: $\forall x,\ y \in R,$ then $x \otimes y \in R$ and $\forall x,\ y \in R,$ then $x \oplus y \in R$

R2: Commutative law: $\forall\ x,\ y \in G;\ \ x \otimes y = y \otimes x$ and $x,\ y \in R;\ \ x \oplus y = y \oplus x$ and (abelian or commutative group).

R3: Associativity: $\forall x,\ y,\ z \in G,$; $x(y \otimes z) = (x \otimes y)z$ and $x(y \oplus z) = (x \oplus y)z$

R4: Distributivity: $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$ (left distributive law)

$(y \oplus z) \otimes x = (y \otimes x) \oplus (z \otimes x)$ (right distributive law)

R5: Existence of identity: There is an element, $e \in R,$ called an identity s.t $x \otimes e = x = e \otimes x\ \forall x \in R,$ and $x \oplus e = x = e \oplus x\ \forall x \in R,$

G6: Existence of inverse: For each $x \in R,$ there is an element $x^{-1} \in R,$ called an inverse of x s.t $x \otimes x^{-1} = e = x^{-1} \otimes x$ and $x \oplus x^{-1} = e = x^{-1} \oplus x$

R1 – R4 show that a ring is an abelian additive group $\langle R, \oplus \rangle$.

A ring obeys associative law under semigroup $\langle R, \otimes \rangle$.

A ring in which $xy = yx$ for every x, y is a commutative ring. In other words, a ring $\langle R, \oplus, \otimes \rangle$ is commutative if $\langle R, \otimes \rangle$ is a commutative semi-group.

An element e of a Ring is called unity (or an identity) of R if $xe = ex = x$ for every $x \in R$.

A non - empty subset S of a ring $\langle R, \oplus, \otimes \rangle$ is called a sub ring of $\langle R, \oplus, \otimes \rangle$ if $\langle S, \oplus, \otimes \rangle$ is also a ring.

A ring may or may not have a unity, however it can be easily shown that if a ring R has an element, $e\ s.t\ xe = ex = x\ \forall\ x \in R$, then e is unique and this e is called the unity or the identity of R. The unity of a ring R is denoted by I.

An element x of a ring is said to be idempotent if $x^2 = x$. A ring R in which every element is idempotent is known as boolean ring.

Definition and Axioms of a Field Algebra

A field $\langle F, \oplus, \otimes \rangle$ is a set defined by two binary compositions $\oplus$ and $, \otimes$. It is otherwise called a commutative division ring. A ring with unity in which all non – zero elements form a group under multiplication is called a division ring. The followings are the axioms of a Field algebra;

F1: $\langle F, \oplus \rangle$ is an abelian (additive) group

F2: $\langle F, \otimes \rangle$ is an abelian (multiplicative) group

F3: $\forall x, y, z \in F,\ x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$ (distributive law)

A non – empty subset S of a field $\langle F \rangle$ is said to be a sub – field of $\langle F \rangle$ if;

(i) $x \in S,\ y \in S \Rightarrow x \oplus y \in S,\ x \otimes y \in S$

(ii) S is a field under the induced $\langle \oplus, \otimes \rangle$ operations

Any subset S of a field $\langle F \rangle$, containing at least two elements is a subfield of $\langle F \rangle$ iff;

(i) $x \in S,\ y \in S \Rightarrow x - y \in S$

(ii) $x \in S,\ y \in S,\ y \neq 0 \Rightarrow (xy)^{-1} \in S$

RESULTS AND DISCUSSION

To obtain the results, we test t- (11, 5, 2)- BIBD with the axioms of the algebraic structures.

Testing 2- (11, 5, 2)- BIBD with axioms of Group Algebra

1. **Multiplicative Operation:** The Cayley table for of Group multiplicative operation is given thus;

Table 2: Caley Table for Multiplicative Binary Operation $(G_E, \otimes)$ Group

$\otimes$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	5	6	10	1	3	5	7	10
3	0	3	6	9	1	4	7	10	2	5	8
4	0	4	8	1	5	9	2	5	10	3	7
5	0	5	10	4	9	3	8	3	7	1	6
6	0	6	1	7	2	8	3	9	4	10	5
7	0	7	3	10	6	2	9	5	1	8	4
8	0	8	5	2	10	7	4	1	9	6	3
9	0	9	7	5	3	1	10	3	6	4	2
10	0	10	9	8	7	6	5	4	3	2	1

Let x_1, x_2 and X_3 be $\in G_E$ and let X_1 = 3, X_2 = 5 and x_3 = 7 , then X's $\in \square_n$

G1: Closure: $\forall x,\ y\ \in G,$ then $x \otimes y \in G$

$x_1 \otimes x_2 \in G_E$; $\forall x_1, x_2 \in \square_{11}$

$3 \otimes 5$ = 15 mod 11 = 4 $\in G_E$ (satisfied)

G2: Associativity: $\forall x,\ y,\ z \in G,$;

$x(y \otimes z) = (x \otimes y)z$

$(x_1 \otimes x_2) \otimes x_3 = x_1 \otimes (x_2 \otimes x_3)$;

$\forall x_1, x_2 \in \square_{11}$

= $(3 \otimes 5) \otimes 7 = 3 \otimes (5 \otimes 7)$

= $15 \otimes 7 = 3 \otimes (35)$

= 105 = 105 = 105 mod 11 = 6 $\in G_E$ (satisfied)

G3: Commutative law: $\forall\ x,\ y \in G;\quad x \otimes y = y \otimes x$

$X_1 \otimes X_2 = X_2 \otimes X_1$; $\forall x_1, x_2 \in \square_{11}$

$3 \otimes 5 = 5 \otimes 3$ = 15 mod 11 = 4 (satisfied)

We can observe from the Cayley table that 3 by 5 = 5 by 3 = 4.

G3: Existence of identity: There is an element, $e \in G,$ called an identity s.t $x \otimes e = x = e \otimes x\ \forall x \in G,$

let 0 be the identity; $3 \otimes 0 = 0$ and $0 \otimes 3 = 0\ \forall x \in G,$

G4: Existence of inverse: For each $x \in G,$ there is an element $x^{-1} \in G,$ called an inverse of x s.t $x \otimes x^{-1} = e = x^{-1} \otimes x$

$3 \otimes -3 \neq 0$ and $-3 \otimes 3 \neq 0$ (not satisfied)

From the table above and result of G4, it is shown that elements do not have identity and inverse element. Hence, $D(G_E)$, that is, t –Steiner Quintuple Designs does not satisfy all the axioms of a Group under multiplicative binary operation hence, not a Group under the multiplicative operation.

2. Additive Binary Operation: The Cayley table for of Group multiplicative operation is given thus;

Table 3: Caley Table for Additive Binary Operation $(G_E, \oplus)$

$\oplus$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10	0
2	2	3	4	5	6	7	8	9	10	0	1
3	3	4	5	6	7	8	9	10	0	1	2
4	4	5	6	7	8	9	10	0	1	2	3
5	5	6	7	8	9	10	0	1	2	3	4
6	6	7	8	9	10	0	1	2	3	4	5
7	7	8	9	10	0	1	2	3	4	5	6
8	8	9	10	0	1	2	3	4	5	6	7
9	9	10	0	1	2	3	4	5	6	7	8
10	10	0	1	2	3	4	5	6	7	8	9

Let x_1, x_2 and X_3 be $\in G_E$ and let x_1 = 3, X_2 = 5 and X_3 = 7 , then X's $\in \square_n$

G1: Closure: $\forall x,\ y\ \in G,$ then $\forall x,\ y\ \in G,$ then $x \oplus y \in G$

$x_1 \oplus x_2 \in G_E$; $\forall X_1, X_2 \in \square_{11}$

$3 \oplus 5$ = 8 mod 11 = 8 $\in G_E$ (satisfied)

G2: Associativity: $\forall x,\ y,\ z \in G,$; then

$x(y \oplus z) = (x \oplus y)z$

$(x_1 \oplus x_2) \oplus x_3 = x_1 \oplus (x_2 \oplus x_3)$;

$\forall x_1, x_2 \in \square_{11}$

= $(3 \oplus 5) \oplus 7 = 3 \oplus (5 \oplus 7)$

= $8 \oplus 7 = 3 \oplus 12$

= 15 = 15 mod 11 = 4 $\in G_E$ (satisfied)

G3: Commutative law: $\forall$ and $x, y \in G$; $x \oplus y = y \oplus x$

$x_1 \oplus x_2 = x_2 \oplus x_1$; $\forall x_1, x_2 \in \square_{11}$

$3 \oplus 5 = 5 \oplus 3$ = 8 mod 11 = 8 (satisfied)

We can observe from the Cayley table that 3 by 5 = 5 by 3 = 4.

G3: Existence of identity: There is an element, $e \in G$, called an identity

s.t $x \oplus e = x = e \oplus x \ \forall x \in G,$

let 0 be the identity;

$3 \oplus 0 = 3$ and $0 \oplus 3 = 3 \ \forall x \in G,$

G4: Existence of inverse: For each $x \in G,$ there is an element $x^{-1} \in G,$ called an inverse of x s.t $x \oplus x^{-1} = e = x^{-1} \oplus x$

$3 \oplus -3 = 0$ and $-3 \oplus 3 = 0$ (satisfied)

From the table above and result of G4, it is shown that elements have identity and inverse element. Hence, $D(G_E)$, that is, t –Steiner Quintuple Designs satisfies all the axioms of a Group under additive binary operation hence, a Group under the additive operation.

Testing 2- (11, 5, 2)- BIBD with axioms of a Ring Algebra

For G_E to be a ring, the design must be abelian additive group and semi-group defined on $(G_E, \otimes, \oplus)$. Also, the two distributive laws must be satisfied.

Let X_1, X_2, X_3 be any three real number in G_E, say $x_1 = 3$, $x_2 = 5$ and $x_3 = 7$ for any $X'_s \in X$

$x_1 \oplus x_2 = x_2 \oplus x_1$ = 8 (mod 11) = 8

$(x_1 \oplus x_2) \oplus x_3 = x_1 \oplus (x_2 \oplus x_3)$
= 4 (mod 11) = 4

There exists $0 \in X$ s.t $x_1 \oplus 0 = 0 \oplus x_1$ = 3

$x_1 \in x, \exists - x_1 \in x$ s.t

$x_1 \oplus (-x_1) = 0$

$(x_1 \otimes x_2) \otimes x_3 = x_1 \otimes (x_2 \otimes x_3)$

$\forall \ x_1, x_2, x_3 \in x$

= 6 mod 11 = 6

$x_1 \otimes (x_2 \oplus x_3) = (x_1 \otimes x_2) \oplus (x_1 \otimes x_3)$

$\forall \ x_1, x_2, x_3 \in X$

= 3 mod 11 = 3

$(x_2 \oplus x_3) \otimes x_1 = (x_2 \oplus x_1) \otimes (x_3 \oplus x_1)$

$\forall \ x_1, x_2, x_3 \in X$

$\Longrightarrow$ 4 mod 11 = 4

$(x_1 \otimes x_2) = x_2 \otimes x_1$

$\Longrightarrow$ 4 mod 11 = $4 \in X$

Testing 2- (11, 5, 2)- BIBD with axioms of a Field Algebra

Let a system be $\langle G_E, \oplus, \otimes \rangle$ be a field, where G_E is a 2- (11,5, 2) , then we test all the axioms of a field on $(G_E, \oplus, \otimes)$.

Let $X_1, X_2, X_3 \in G_E$, then $\langle G_E, \oplus \rangle$ is closed under addition and $\langle G_E, \otimes \rangle$ is closed under multiplication. Given x_1 = 2, x_2 = 4 and x_3 = 6, then

i. $x_1 \oplus x_2 \oplus x_3 \in X$ and $x_1 \otimes x_2 \otimes x_3 \in G_E$ are closed
12 (mod 11) and 48 (mod 11)
$\Rightarrow 1 \in X$ and $4 \in G_E$

ii. $(x_1 \oplus x_2) \oplus x_3 = x_1 \oplus (x_2 \oplus x_3)$ and $(x_1 \otimes x_2) \otimes x_3 = x_1 \otimes (x_2 \otimes x_3)$
$\Rightarrow 1 \in X$ and $4 \in G_E$, G_E is associative under $(x, \oplus)$

iii. $x_1 \oplus x_2 = x_2 \oplus x_1$ = 6 mod 11 $\Rightarrow$ $6 \in G_E$
$x_1 \otimes x_2 = x_2 \otimes x_1$ = 8 mod 11 $\Rightarrow$ $8 \in G_E$
$\Rightarrow$ X is commutative under $(x, \otimes)$

iv. There exists $0 \in X$, s.t $x_i \oplus 0 = 0 \oplus x_i = x_i, \forall \in G_E$
$\Rightarrow$ $2 \in X$
There exists $1 \in X$; s.t $x_1 \otimes 1 = 1 \otimes x_1 = x_1$; $\forall x_1 \in X$
From table 4.3, at least 1 $\notin X$ s. t $x_1 \otimes 1 = 1 \otimes x_1 \neq x_1$; $\forall x_1 \in G_E$

v. There exists $-x_1 \in G_E$; s.t $x_1 \oplus (-x_1) = -x_1 \oplus x_1 = 0$; $\forall x_1 \in G_E$
$\forall x_1 \neq 0 \in X$, $\exists$ $x_1^{-1} \in X$, s.t
$x_1 \otimes x_1^{-1} = x_1^{-1} \otimes x_1 = 1$

vi. $x_1 \oplus (x_2 \otimes x_3) = (x_1 \otimes x_2) \oplus (x_1 \otimes x_3)$ (holds)
$\Rightarrow$ 9 mod 11 = 9 $\in X$

Therefore, $\langle X, \oplus \rangle$ is an abelian group but in $(X, (0), \otimes)$, some of the axioms of a field are not satisfied. To this end, 2- (11, 5, 2) does not form a field.

Discussion

A t- Steiner quintuple designs 2-(11, 5, 2)) represented as G_E was tested with the axioms of Group, Ring and Field algebra respectively in order to ascertain its algebraic structure. The results showed that the design did not satisfy all the axioms of the multiplicative binary operation but satisfied that of the additive binary operation. Furthermore, result showed that the t-Steiner quintuple design satisfied all the axioms of a ring algebra but failed to satisfy all the axioms of a field algebra.

CONCLUSION

The algebraic structure of a t- Steiner quintuple system of balanced incomplete block design can be summarized thus;

i. It is a group algebra under the additive binary operation
ii. It is not a group algebra under the multiplicative binary operation

iii. It is a ring and hence a semi- group, commutative semi-ring and commutative ring.
iv. It is not a field algebra.

In conclusion, a t-Steiner quintuple design of BIBD is a Group algebra under the additive group binary operation and a Ring algebra. It is neither a Group under the multiplicative operation nor a Field algebra.

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