FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645-2944
Vol. 9 No. 11, November 2025, pp 55 - 68
DOI: https://doi.org/10.33003/fjs-2025-0911-3966

MODELING THE DYNAMICS OF DIPHTHERIA WITH PUBLIC AWARENESS

[*1,2]Abbas Abubakar, [1,3]Aliyu Ahmad, [4]Sani Musa, [1]Ya,u U Ahmad, [2]Aliyu Abba, [5]Lawan Abdullahi, [6]Jibrin Y Musa and [7]Wakili Abubakar.

[1]Department of Mathematics, Federal University Dutse, Jigawa, Nigeria
[2]Department of Statistics, Jigawa State Polytechnic Dutse, Jigawa, Nigeria.
[3]Department of Mathematics, Jigawa State College Education and Legal Studies Ringim,
[4]Department of Mathematics, Sule Lamido University Kafin Hausa, Jigawa,, Nigeria.
[5]Department of Information Technology, Sule Lamido University Kafin Hausa,
[6]Department of Foundation Courses and Remedial Studies,
Binyaminu Usman Polytechnic, Hadejia
[7]Department of Computer Science, Jigawa State College of Remedial
and Advanced Studies, Babura
[1]Corresponding authors' email: abbas.abubakar19@jigpoly.edu.ng

Abstract

Diphtheria is a highly contagious bacterial infection that primarily affects the mucous membranes of the nose, throat, and airways. It is caused by the bacterium Corynebacterium diphtheriae, which produces a toxin that damages the respiratory system, heart, and nervous system. Despite the existence of effective vaccines, diphtheria continues to pose a threat to global health. In this paper, we developed a nonlinear deterministic model which incorporates public awareness and isolation to describe the dynamics of diphtheria. Analysis of the model reveals that the boundedness and positivity of solutions have been ascertained, diphtheria free equilibrium is both locally and globally asymptotically stable whenever the associated control reproduction number $\mathcal{R}_c < 1$ and unstable when $\mathcal{R}_c > 1$, similarly the endemic equilibrium is globally asymptotically stable when the control reproduction $\mathcal{R}_c > 1$ and $\phi = \tau = \delta_1 = \delta_2 = 0$. Moreover, the model undergoes backward bifurcation in which a stable disease-free equilibrium coexists with a stable endemic equilibrium. The epidemiological implication of backward bifurcation is $\mathcal{R}_c < 1$ is necessary but not sufficient condition for control of diphtheria even when the classical requirement are satisfied. The most sensitive parameters for the control of the spread of diphtheria are identified by forward sensitivity index method and found that contact rate β and progression rate of exposed individuals to infected compartment are the most sensitive parameters for increasing the transmission. On the contrary, isolation rate τ and recovery rate of infected individuals γ_1 are the most sensitive for reducing the spread. Furthermore, the numerical simulation performed shows the impact of public awareness and isolation in controlling the spread of diphtheria. Finally, the result shows public awareness will help in curtailing the spread of diphtheria infection, and when isolation is applied on infected individuals.

Keywords: Diphtheria; Awareness; Isolation; Reproduction number; Stability analysis.

INTRODUCTION

Diphtheria, an infection of the respiratory tract caused by Corynebacterium diphtheriae, is a highly contagious and potentially deadly disease (Johnson *et al.,* 2024). Diphtheria primarily affects the respiratory system, but can also affect other parts of the body, resulting in severe complications and potentially fatal consequences if left untreated (Egbune *et al.,* 2024). Diphtheria can spread easily from one person to another by coughing or sneezing, making it highly contagious. Some individuals may carry the bacteria and spread it to others without even showing symptoms themselves, while others may develop mild symptoms or, in severe cases, experience life-threatening complications (WHO 2024). The symptoms of diphtheria usually develop within 2-5 days after exposure to the bacteria, with initial symptoms often appearing mild (Egbune *et al.,* 2024). The symptoms of diphtheria include fever, exhaustion, cyanosis, sore throat, trouble swallowing, breathing, and sometimes paralysis (Musa *et al.,* 2024). In severe cases of diphtheria, the disease can ravage healthy respiratory tissue, leading to the formation of a thick, gray coating known as "Pseudomembrane" that can cause significant damage to the heart and kidneys, potentially resulting in death if left untreated (Musa *et al.,* 2025).

In the early 20th century, before the advent of the diphtheria Toxoid vaccine in 1923, diphtheria was a devastating disease that claimed countless lives, with children being particularly vulnerable to its lethal effects (Musa *et al.,* 2025). Despite the existence of effective vaccines, diphtheria continues to pose a threat to global health. The DTaP vaccine, which provides robust protection against diphtheria, pertussis and tetanus, is a key element in safeguarding public health against these highly contagious and potentially deadly diseases (Johnson *et al.,* 2024). While vaccination efforts in the mid-20th century successfully curbed the disease, its resurgence in recent years, particularly in developing nations, can be attributed to declining vaccination rates and waning immunity among adults, underscoring the importance of maintaining high vaccination coverage to protect against this deadly disease (Sahib *et al.,* 2024).

The World Health Organization (WHO) has recently reported an alarming increase in diphtheria cases worldwide, with regions experiencing limited healthcare infrastructure as the most impacted (Sahib *et al.,* 2024). The number of reported cases in 2019 (22,986) reached a level not seen since 1996, highlighting diphtheria as a persistent public health concern (Pembroke *et al.,* 2023). The African region, particularly Nigeria, has been the epicenter of the diphtheria outbreak, with the majority of reported cases to the WHO originating from this region in 2019 (Traugott *et al.,* 2023). The situation has continued to worsen in Africa, with a total of 40,929 suspected cases and 828 deaths recorded in seven countries, with Nigeria accounting for the highest number of cases and fatalities (WHO, 2024). Although the WHO European region considers diphtheria an uncommon occurrence, cases have been reported in this region between 2012 and 2021, amounting to 452 out of the over 96 000 cases worldwide (WHO, 2023). The Southeast Asia region, with India, Nepal, and Indonesia being particularly affected, accounts for 65.3% of the global diphtheria burden, reporting an average of 8,243 cases annually to the WHO, demonstrating the prevalence of this disease in the region (Elsinga *et al.,* 2023).

Many mathematical models have been developed in studying the dynamics of infectious diseases, which include diphtheria. Some of these models are (Andrawus *et al.,* (2025) introduced a deterministic mathematical model that takes into account the often-neglected factors of awareness and surveillance in the transmission dynamics of diphtheria. Both their theoretical and numerical findings demonstrated that, with an increase in public awareness and efficient surveillance systems, diphtheria can be successfully eradicated within a span of 10 years. (Izzati *et al.,* (2024) conducted a study that examined the impact of vaccination completeness on the dynamics of diphtheria spread, titled "Dynamical analysis of a mathematical model on the spread of diphtheria disease with vaccination completeness factors." The results of their study demonstrated that vaccination completeness plays a significant role in shaping the population's response to diphtheria outbreaks as modeled. These findings highlight the importance of considering vaccination completeness as a key factor in strategies to prevent the re-emergence of diphtheria. (Egbune *et al.,* 2024) conducted a study titled "Mathematical Analysis of Spread and Control of Diphtheria with Emphasis on Diphtheria Antitoxin Efficiency" to investigate the spread and control of diphtheria with a particular focus on the efficacy of Diphtheria Antitoxin. Their findings showed that Diphtheria Antitoxin efficiency plays a significant role in reducing the burden of the disease, preventing severe cases, and limiting the spread of outbreaks. (Johnson *et al.,* 2024) employed mathematical modeling techniques to investigated the spread of diphtheria and the effectiveness of various control measures. Their findings suggested that the most effective strategy for reducing the impact of diphtheria is to increase vaccination rates at birth, as this leads to a higher number of individuals recovering from the disease. They also found that quarantine measures are effective in limiting the spread of diphtheria, and maternal derived immunity, achieved through vaccination during pregnancy, holds great promise in protecting against the disease. (Musa *et al.,* (2025) conducted a study to examine the dynamics of the diphtheria epidemic in Nigeria, focusing on the Kano State outbreak. Their findings indicated that implementing public awareness campaigns and isolating infected individuals could significantly reduce the spread of diphtheria within affected communities. The results of their study highlighted the importance of effective communication and containment strategies in controlling diphtheria outbreaks, particularly in regions with limited access to healthcare resources.

In light of the aforementioned researchers results, we developed a mathematical model to assess the impact of awareness and isolation on diphtheria dynamics, motivated by the work of (Musa *et al.,* 2025) by incorporating aware and unaware susceptible compartments, which is crucial for understanding the diphtheria dynamics, we also consider awareness as a rate, not the same as in (Musa *et al.,* 2025).

MODEL DESCRIPTION

The model is formulated to study the transmission dynamics of diphtheria infection. The total population denoted by $N(t)$ is divided into six disjoint compartments. Susceptible unaware $S_u(t)$, susceptible aware $S_a(t)$, exposed $E(t)$, infected $I(t)$, isolated $J(t)$ and the removed or recovered compartment $R(t)$. The recruitment into susceptible class is by birth and migration at a constant rate π. p is the fraction of recruited individuals that are aware. Progression from susceptible unaware into susceptible aware ϕ compartment is through public awareness. The susceptible population decrease with the emergence of the infection at the rate λ. Unaware are more susceptible to become infected (this is the reason for adding the parameter θ reducing the rate of infection in aware). Exposed individuals progress to infected compartment at rate σ. Infected individuals may either progress to isolated compartment τ or moved to recovered compartment γ_1. The mortality due to disease δ_1, δ_2 only occurs in infected and isolated compartments, while the natural death occurs in all the six compartments μ. The total population will therefore be

$$N(t) = S_u(t) + S_a(t) + E(t) + I(t) + J(t) + R(t).$$

$$\begin{aligned}
\frac{dS_u}{dt} &= \pi(1-p) - (\lambda + \phi + \mu)S_u,\\
\frac{dS_a}{dt} &= \pi p + \phi S_u - (\theta\lambda + \mu)S_a,\\
\frac{dE}{dt} &= \lambda(S_u + \theta S_a) - (\sigma + \mu)E,\\
\frac{dI}{dt} &= \sigma E - (\tau + \gamma_1 + \mu + \delta_1)I,\\
\frac{dJ}{dt} &= \tau I - (\gamma_2 + \mu + \delta_2)J,\\
\frac{dR}{dt} &= \gamma_1 I + \gamma_2 J - \mu R.
\end{aligned} \tag{1}$$

Where

$$\lambda = \frac{\beta I}{N}$$

THEORETICAL ANALYSIS OF THE MODEL

Boundedness and Positivity

The solution of the model system (1) is constrained within a manifold or space Ω, denoted by

$$\begin{aligned}
\Omega = \{(&S_u(t), S_a(t), E(t), I(t),\\
&J(t), R(t)) \;\epsilon\; R_+^6 : N \leq \frac{\pi}{\mu}.
\end{aligned} \tag{2}$$

Theorem 1 *The region Ω is positively invariant and an attractor.*

Proof 1 *Our goal is to demonstrate that R_+^6 is positively invariant, meaning that all solutions to system (1) that begin within Ω remain within Ω at all times. Assume that $R(0)) > 0$ and that $S_u(0), S_a(0), E(0), I(0), and J(0) > 0$. If $S_u(0)$ and $S_a(0)$ are not both positive, then $S_u(t) > 0$ and $S_a(t) > 0$ for $t \in [0, \tilde{t})$ exist at some time $\tilde{t} > 0$ and $S_u(\tilde{t}) = S_a(\tilde{t}) = 0$. Using system (1) third, fourth, and fifth equations, we now get,*

$$\begin{aligned}
\frac{dE(t)}{dt} &\geq -(\mu + \sigma)E(t) \quad \textit{for } t \in [0, \tilde{t}),\\
\frac{dI(t)}{dt} &\geq -(\mu + \delta_1 + \tau + \gamma_1)I(t) \quad \textit{for } t \in [0, \tilde{t}),\\
\frac{dJ(t)}{dt} &\geq -(\mu + \delta_2 + \gamma_2)J(t) \quad \textit{for } t \in [0, \tilde{t}),
\end{aligned} \tag{3}$$

Thus, $E(0) > 0, I(0) > 0$ and $J(0) > 0$ for $t \in [0, \tilde{t})$. As a result, using the system (1) first and second equations, we've obtained

$$\begin{aligned}
\frac{dS_u(t)}{dt} &\geq -(\phi + \mu + \lambda)S_u(t) \quad \textit{for } t \in [0, \tilde{t}),\\
\frac{dS_a(t)}{dt} &\geq -(\mu + \theta\lambda)S_a(t) \quad \textit{for } t \in [0, \tilde{t}).
\end{aligned}$$

One can see that, $S_u(0) > 0$ and $S_a(0) > 0$ which contradict our assumption of $S_u(\tilde{t}) = S_a(\tilde{t}) = 0$. Hence $S_u(t)$ and $S_a(t)$ are positive. Alternatively, we can consider a subsystem of (1) excluding the first and second equations, which can be expressed as a matrix form, providing a clear demonstration of the positivity of the remaining state variables in the model.

$$\frac{dX(t)}{dt} = \mathcal{M}Y(t) + B(t), \tag{4}$$

with

$$Y(t) = \begin{pmatrix} E, & I, & J, & R \end{pmatrix}^T,$$

$$\mathcal{M} = \begin{pmatrix} -k_2 & m & 0 & 0\\ \sigma & -k_3 & 0 & 0\\ 0 & \tau & -k_4 & 0\\ 0 & \gamma_1 & \gamma_2 & -\mu \end{pmatrix}, \tag{5}$$

$$B(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T,$$

where, $m = \beta\frac{S_u + \theta S_a}{N}$, $k_2 = \mu + \sigma$, $k_3 = \mu + \delta_1 + \tau + \gamma_1$ and $k_4 = \mu + \delta_2 + \gamma_2$, The fact that both $S_u(t)$ and $S_a(t)$ are non-negative indicates that $\mathcal{M}$ is a Metzler matrix. indicating that subsystem (4) is a monotone system (Ibrahim et al., 2025). Therefore, under the flow of subsystem (4), R_+^4 is invariant. R_+^6 consequently becomes positively invariant under the system's flow (1).

Diphtheria Free Equilibrium Point

The model system (1) has a diphtheria-free equilibrium ϵ^0 when the community is depleted of diphtheria. By solving the noninfected classes and setting the infection classes and the right-hand sides of equation (1) to zero, one can mathematically determine this equilibrium.

$$\epsilon^0 = (S_u^0, S_a^0, E^0, I^0, J^0, R^0) \tag{6}$$

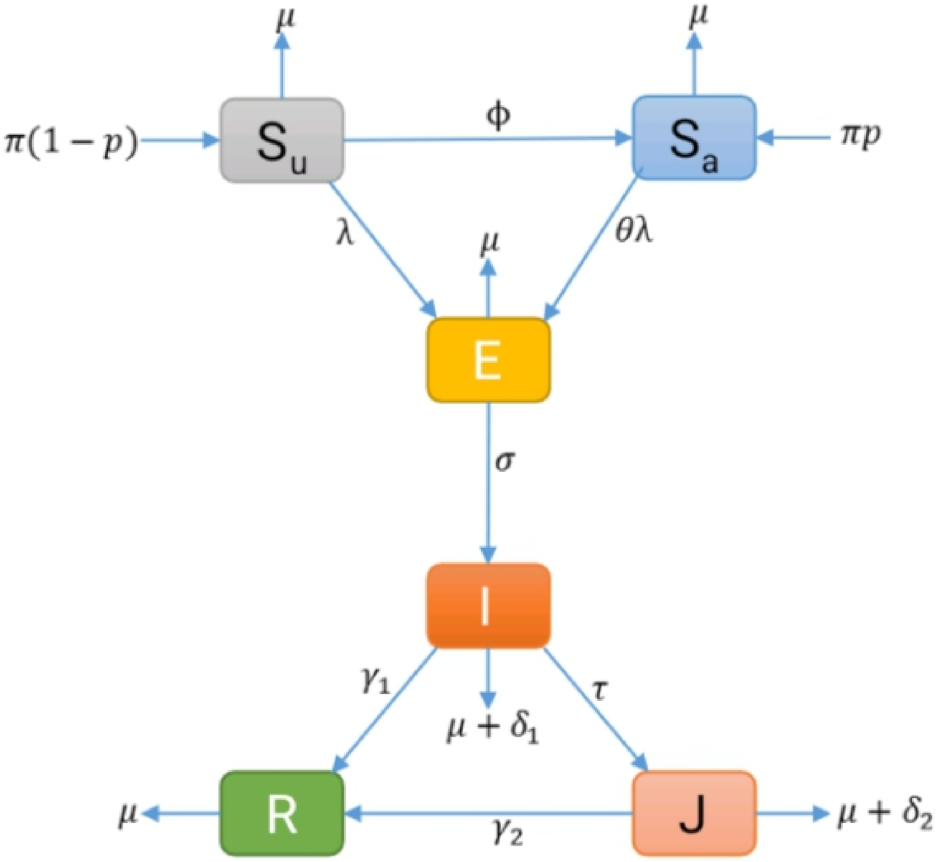

Figure 1: Schematic diagram of the model (1). Solid arrows indicate transitions and expressions next to arrows show the per ca-pita flow rate between compartments.

Table 1: Interpretation of the state variables and parameters used in the model (1).

Variable	Description
N	Total population
S_u	Susceptible unaware individuals
S_a	Susceptible aware individuals
E	Exposed individuals
I	Infected individuals
J	Isolated individuals
R	Recovered individuals
Parameter	**Description**
π	Recruitment rate of susceptible individuals
p	A proportion of individuals who are aware
ϕ	Awareness rate
μ	Natural mortality rate
β	Effective contact rate
θ	Reduction risk of infection
σ	progression rate
τ	Isolation rate
γ_1	Recovery rate of infected individuals
γ_2	Recovery rate of isolated individuals
δ_1, δ_2	Diphtheria induced death rate

$$S_u^0 = \frac{(1-p)\pi}{\phi + \mu} \qquad (7)$$

$$S_a^0 = \frac{\pi((1-p)\phi + p(\phi+\mu))}{\mu(\phi+\mu)} \tag{8}$$

$$(E^0, I^0, J^0, R^0) = (0,0,0,0) \tag{9}$$

Basic Reproduction Number

According to (Abubakar *et al.*,2025, Andrawus *et al.*, 2025 and Ibrahim *et al.*, 2025) , the next generation operator technique was used to find the fundamental reproduction number, $\mathcal{R}_0 = \rho(\mathcal{F}_1\mathcal{F}_2^{-1})$. The matrices $\mathcal{F}_1$ present the new infection terms and $\mathcal{F}_2$ for the remaining transition terms are obtained as follows:

$$\mathcal{F}_1 = \begin{bmatrix} 0 & \frac{\beta\left(S_u^0+\theta\, S_a^0\right)}{N^0} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{F}_2 = \begin{bmatrix} \sigma+\mu & 0 & 0 \\ -\sigma & \tau+\gamma_1+\mu+\delta_1 & 0 \\ 0 & -\tau & \gamma_2+\mu+\delta_2 \end{bmatrix} \tag{10}$$

and

$$\mathcal{F}_2^{-1} = \begin{bmatrix} (\sigma+\mu)^{-1} & 0 & 0 \\ \frac{\sigma}{(\sigma+\mu)(\tau+\gamma_1+\mu+\delta_1)} & (\tau+\gamma_1+\mu+\delta_1)^{-1} & 0 \\ \frac{\sigma\,\tau}{(\sigma+\mu)(\tau+\gamma_1+\mu+\delta_1)(\gamma_2+\mu+\delta_2)} & \frac{\tau}{(\tau+\gamma_1+\mu+\delta_1)(\gamma_2+\mu+\delta_2)} & (\gamma_2+\mu+\delta_2)^{-1} \end{bmatrix} \tag{11}$$

$$\mathcal{F}_1\mathcal{F}_2^{-1} = \begin{bmatrix} \frac{\beta\left(S_u^0+\theta\, S_a^0\right)\sigma}{N^0(\sigma+\mu)(\tau+\gamma_1+\mu+\delta_1)} & \frac{\beta\left(S_u^0+\theta\, S_a^0\right)}{N^0(\tau+\gamma_1+\mu+\delta_1)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

The eigenvalues of the matrix $\mathcal{F}_1\mathcal{F}_2^{-1}$ are found using the $\det(\mathcal{F}_1\mathcal{F}_2^{-1} - MI) = 0$ The eigenvalues are represented by M in this case. (13) computes the eigenvalues in this way:

$$\begin{bmatrix} 0 \\ 0 \\ \frac{\beta\left(S_u^0+\theta\, S_a^0\right)\sigma}{N^0(\sigma+\mu)(\tau+\gamma_1+\mu+\delta_1)} \end{bmatrix}. \tag{13}$$

The dominant eigen value from (13) gives the control reproduction number

$$\mathcal{R}_c = \frac{\beta\sigma(S_u^0+\theta S_a^0)}{N^0(\sigma+\mu)(\tau+\gamma_1+\mu+\delta_1)} \tag{14}$$

Substituting S_u^0, S_a^0 and N^0 in (14) we have

$$\mathcal{R}_c = \frac{\beta\sigma[(1-p)(\mu+\theta\phi)+\theta p(\phi+\mu)]}{(\sigma+\mu)(\tau+\gamma_1+\mu+\delta_1)} \tag{15}$$

Interpretation of $\mathcal{R}_c$
The control reproduction number $\mathcal{R}_c$ is the number of new infections produced by diphtheria-infected individuals in a population consisting of susceptible aware and unaware individuals. In a precise notion, it is the number produced by an infected individual in presence of controls in a community.

When there is no awareness and isolation in the society (i.e $\phi = \tau = 0$) we obtained the basic reproduction number as

$$\mathcal{R}_0 = \frac{\beta\mu\sigma[(1-p)+\theta p]}{(\sigma+\mu)(\gamma_1+\mu+\delta_1)} \tag{16}$$

Interpretation of $\mathcal{R}_0$
The basic reproduction number $\mathcal{R}_0$ is the number of new infections produced by diphtheria-infected individuals in a fully susceptible population. In a precise notion, it is the number produced by an infected individual in the absence of awareness and isolation in the community.

Local Asymptomatic Stability of Diphtheria Free Equilibrium

A system is said to be locally asymptotically stable if it remains stable even after experiencing a small disturbance. This means that a diphtheria-free equilibrium that is locally asymptotically stable represents a situation where a small number of infections will not lead to a widespread outbreak. Mathematically, this condition is met if all eigenvalues of the linearized system (1) have a negative real part. Therefore, we have the following theorem:

Theorem 2 *The Diphtheria free equilibrium ϵ^0 of the model (1) ϵ^0 is lobally-asymptotically stable (GAS) in Ω if the control reproduction number $\mathcal{R}_c < 1$, and unstable if $\mathcal{R}_c > 1$.*

Proof 2 *The linearization of system (1) and the computation of the Jacobian matrix at the diphtheria-free equilibrium can be done by performing the following steps:*

$$J(\epsilon^0) = \begin{bmatrix} -\mu-\phi & 0 & 0 & -\beta M & 0 & 0 \\ \phi & -\mu & 0 & -\beta N & 0 & 0 \\ 0 & 0 & -\mu-\sigma & \beta(M+N) & 0 & 0 \\ 0 & 0 & \sigma & -\tau-\gamma_1-\mu-\delta_1 & 0 & 0 \\ 0 & 0 & 0 & \tau & -\gamma_2-\mu-\delta_2 & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & -\mu \end{bmatrix}, \tag{17}$$

where

$$M = \frac{\mu(1-p)}{\phi+\mu}, N = \frac{\mu((1-p)\phi+p(\phi+\mu))}{\mu(\phi+\mu)}$$

reducing equation (17) into row echelon yield

$$J(\epsilon^0) = \begin{bmatrix} -\mu-\phi & 0 & 0 & -\beta M & 0 & 0 \\ 0 & -\mu & 0 & \frac{(-\beta N-\beta M)\phi-\beta N\mu}{\mu+\phi} & 0 & 0 \\ 0 & 0 & -\mu-\sigma & \beta(M+N) & 0 & 0 \\ 0 & 0 & 0 & \frac{\sigma\beta(M+N)-(\mu+\sigma)(\mu+\tau+\delta_1+\gamma_1)}{\mu+\sigma} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma_2-\mu-\delta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu \end{bmatrix} \tag{18}$$

Using the Maple software, the eigenvalues are determined as follows: (21)

$$\begin{bmatrix} -\gamma_2-\mu-\delta_2 \\ -\mu-\sigma \\ \frac{\sigma\beta(M+N)-(\mu+\sigma)(\mu+\tau+\delta_1+\gamma_1)}{\mu+\sigma} \\ -\mu-\phi \\ -\mu \\ -\mu \end{bmatrix} \tag{19}$$

Clearly, $\lambda_1, \lambda_2, \lambda_4, \lambda_5$ and λ_6 are all negatives from (19).
Then λ_3 is also negative

$$\iff \frac{\sigma\beta(M+N)-(\mu+\sigma)(\mu+\tau+\delta_1+\gamma_1)}{\mu+\sigma} < 0, \tag{20}$$

$$\iff \sigma\beta(M+N)-(\mu+\sigma)(\mu+\tau+\delta_1+\gamma_1) < 0,$$

$$\iff \beta\sigma(M+N) < (\mu+\sigma)(\mu+\delta_1+\tau+\gamma_1), \tag{22}$$

$$\iff \frac{\beta\sigma(M+N)}{(\mu+\sigma)(\mu+\delta_1+\tau+\gamma_1)} < 1, \tag{23}$$

substituting M and N in 23 λ_3 is also negative

$$\iff \frac{\beta\sigma[(1-p)(\mu+\theta\phi)+\theta p(\phi+\mu)]}{(\sigma+\mu)(\mu+\delta_1+\tau+\gamma_1)} = \mathcal{R}_c < 1. \tag{24}$$

The analysis shows that if $\mathcal{R}_c < 1$, then all eigenvalues are negative, ensuring the stability of the disease-free equilibrium point. Conversely, if $\mathcal{R}_c > 1$, the eigenvalues are unstable, indicating that the disease will continue to spread in the population. This completes the proof of Theorem (2)..

Interpretation of theorem (2)

Theorem (2) epidemiologically demonstrates that a community will remain free of endemic diphtheria despite a small number of infected individuals if the control reproduction number $\mathcal{R}_c$ is less than 1. This implies that if the number of diphtheria cases is low and $\mathcal{R}_c$ is kept below 1, the disease can be effectively controlled and prevented from becoming endemic.

Global Asymptomatic Stability of Diphtheria Free Equilibrium

Theorem 3 *The Diphtheria free equilibrium ϵ^0 of the model (1) ϵ^0 is globally-asymptotically stable (GAS) in Ω if the control reproduction number $\mathcal{R}_c < 1$, and unstable if $\mathcal{R}_c > 1$.*

Proof 3 *The proof of the theorem hinges on ensuring that conditions (P_1) and (P_2) as in (Castillo-Charez and Son, 2004) hold true when $\mathcal{R}_c < 1$. The model (1) can be written in the following form:*

$$\frac{dP_1}{dt} = F(P_1, P_2), \tag{25}$$

$$\frac{dP_2}{dt} = G(P_1, P_2); G(P_1, 0) = 0, \tag{26}$$

where $P_1 = (S_u^0, S_a^0, R^0)$ and $P_2 = (E^0, I^0, J^0)$, where $P_1 \in R^3_+$ is denoting the uninfected population and $P_2 \in R^3_+$ denoting the infected population. The diphtheria free equilibrium is now denoted as, $M^0 = (P_1^*, 0)$

where,

$P_1^* = (N^0, 0)$ Now for the first condition, globally asymptotic stability of P_1^*, gives

$$\frac{dP_1}{dt} = F(P_1, 0) = \begin{bmatrix} \pi(1-p) - (\phi+\mu)S_u^0 \\ \pi p + \phi S_u - \mu S_a^0 \\ 0 \end{bmatrix}. \tag{27}$$

A linear ODE solving gives,

$$S_u^0(t) = \frac{\pi(1-p)}{(\phi+\mu)} - \frac{\pi(1-p)}{(\phi+\mu)}e^{-(\phi+\mu)t} + S_u^0(0)e^{-(\phi+\mu)t}, \tag{28}$$

$$S_a^0(t) = \frac{\pi p + \phi S_u^0}{\mu} - \frac{\pi p + \phi S_u^0}{\mu}e^{-\mu t} + S_a^0(0)e^{-\mu t}. \tag{29}$$

Now, clearly from system (1) we have, $S_u^0(t) + S_a^0(t) + R^0(t) \to N^0(t)$ as $t \to \infty$ regardless of the value of $S_u^0(t), S_a^0(t)$ and $R^0(t)$. Thus , $P_1^* = (N^0, 0)$ is globally asymptotically stable.

Next, for the second condition, that is $\tilde{G}(P_1, P_2) = AP_2 - G(P_1, P_2) \geq 0$

$$A = \begin{pmatrix} -(\mu+\sigma) & \frac{\beta(S_u+\theta S_a)}{N} & 0 \\ \sigma & -(\mu+\delta_1+\tau+\gamma_1) & 0 \\ 0 & \tau & -(\gamma_2+\mu+\delta_2) \end{pmatrix}. \tag{30}$$

The matrix A is a Metziller matrix (the off-diagonal elements of are non negative).

$$G(P_1, P_2) = \begin{pmatrix} \frac{\beta I^0}{N^0}S_u^0 + \frac{\theta\beta I^0}{N^0}S_a^0 - (\sigma+\mu)E^0 \\ \sigma E^0 - (\mu+\delta_1+\tau+\gamma_1)I^0 \\ \tau I - (\mu+\delta_2+\gamma_2)J^0 \end{pmatrix} \tag{31}$$

Then,

$$\tilde{G}(P_1, P_2) = AP_2 - G(P_1, P_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{32}$$

That is

$$\tilde{G}(P_1, P_2) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T. \tag{33}$$

It is obvious that $\tilde{G}(P_1, P_2) = 0$.

Diphtheria Endemic Equilibrium Point

A diphtheria reaches endemic equilibrium when it has spread and remained in a community for an extended period of time. We may show that the equilibrium state exists by figuring out the implicit solutions of the model (1) state variables in terms of λ, even though they are hard to solve explicitly. Given that the variables in the condition of endemic equilibrium are $S_u^{**}, S_a^{**}, E^{**}, I^{**}, J^{**}, R^{**}$ and that the force of infection is λ^{**}, the solutions of the state variables are as follows:

$$\begin{aligned} S_u^{**} &= \frac{(1-p)\pi}{\lambda+k_1}, \\ S_a^{**} &= \frac{\pi((1-p)\phi + p(\lambda+k_1)}{(\theta\lambda+\mu)(\lambda+k_1)}, \\ E^{**} &= \frac{\lambda\pi((1-p)(\theta\lambda+\mu+\theta\phi) + \theta p(\lambda+k_1))}{k_2(\lambda+k_1)(\theta\lambda+\mu)}, \\ I^{**} &= \frac{\lambda\pi\sigma((1-p)(\theta\lambda+\mu+\theta\phi) + \theta p(\lambda+k_1))}{k_2k_3(\lambda+k_1)(\theta\lambda+\mu)}, \\ J^{**} &= \frac{\lambda\pi\tau\sigma((1-p)(\theta\lambda+\mu+\theta\phi) + \theta p(\lambda+k_1))}{k_2k_3k_4(\lambda+k_1)(\theta\lambda+\mu)}, \\ R^{**} &= \frac{\lambda\pi\sigma\gamma_1((1-p)(\theta\lambda+\mu+\theta\phi) + \theta p(\lambda+k_1))}{\mu k_2k_3(\lambda+k_1)(\theta\lambda+\mu)} \\ &\quad + \frac{\lambda\pi\tau\sigma\gamma_2((1-p)(\theta\lambda+\mu+\theta\phi) + \theta p(\lambda+k_1))}{\mu k_2k_3k_4(\lambda+k_1)(\theta\lambda+\mu)}. \end{aligned} \tag{34}$$

Where

$k_1 = \phi+\mu, k_2 = \sigma+\mu, k_3 = \tau+\gamma_1+\mu+\delta_1$ and $k_4 = \gamma_2+\mu+\delta_2$

Existence of Diphtheria Endemic Equilibrium Point

To verify the existence of the endemic equilibrium point of the proposed model (1), the Descartes rule of sign was applied. This rule states that in a polynomial equation with real coefficients and degree $n \geq 2$, the number of positive roots is equal to the number of changes in the sign of the terms in the equation, or less than this by an even number. In this case, the force of infection at the endemic state was represented by:

$$\lambda^{**} = \frac{\beta I^{**}}{N^{**}} \tag{35}$$

where

$$N^{**} = S_u^{**} + S_a^{**} + E^{**} + I^{**} + J^{**} + R^{**} \tag{36}$$

When (34) is substituted into (35), the following quadratic equation in terms of λ^* is obtained: When $\mathcal{R}_c < 1$, the disease-free equilibrium of equation (6) is equivalent to $\lambda^{**} = 0$

$$\lambda^{**2} Q_1 + \lambda^{**} Q_2 + Q_3 = 0, \tag{37}$$

where

$$\begin{aligned}
Q_1 &= (1-p)(k_4+\tau)\mu\theta\sigma + (1-p)(\gamma_1 k_4 + \tau\gamma_2) + (1-p)\mu k_3 k_4 \\
&\quad + (k_4+\tau)\mu\theta\sigma p + (\gamma_1 k_4 + \tau\gamma_2)\theta\sigma p + \mu\sigma k_3 k_4, \\
Q_2 &= (1-p)\mu\theta k_2 k_3 k_4 + (1-p)\mu\theta\phi k_3 k_4 + (1-p)\mu\theta\phi\sigma k_4 + (1-p)\mu\theta\sigma\phi\tau \\
&\quad + (1-p)\mu\sigma\gamma_1 k_4 + (1-p)\theta\phi\sigma\gamma_1 + (1-p)\mu\sigma\tau\gamma_2 + (1-p)\theta\phi\sigma\gamma_2\tau + (k_2+\theta)\mu k_3 k_4 \\
&\quad + (k_4+\tau)\mu\sigma\theta p k_1 + (\gamma_1 k_4 + \tau\gamma_2)\theta\sigma p k_1 + (1-p)(k_4+\tau)\mu - \beta\mu\theta\sigma(1+p), \\
Q_3 &= (\phi+\mu)(\sigma+\mu)(\tau+\gamma_1+\mu+\delta_1)[1-\mathcal{R}_c].
\end{aligned} \tag{38}$$

The number of endemic equilibrium points in the system is equivalent to the number of positive roots of equation (37). It is important to note that the number of roots of equation (37) is equal to the number of changes in sign of the constants Q_1, Q_2, and Q_3, as discussed in (Andrawus *et al* 2024). It is evident that Q_1 is positive since all the parameters are positive and $0 < p < 1$, which implies from equations (37) and (38) the following theorem:

Theorem 4 *The model system (1) has positive endemic equilibrium:*

i. If $Q_2 > 0$ and $Q_3 > 0 \iff \mathcal{R}_c < 1$, implies that the system has no positive equilibrium.
ii. If $Q_2 < 0$ and $Q_3 < 0 \iff \mathcal{R}_c > 1$, implies that the system has unique positive equilibrium.
iii. If $Q_2 > 0$ and $Q_3 < 0 \iff \mathcal{R}_c > 1$, implies that the system has unique positive equilibrium.
iv. If $Q_2 < 0$ and $Q_3 > 0 \iff \mathcal{R}_c < 1$ and $Q_2^2 - 4Q_1Q_3 > 0$, implies that the system has two positive equilibria.

Case (ii) and (iii) of theorem (4) were used as a basis for the following theorem.

Theorem 5 *A unique positive endemic equilibrium exists in the system (1) if $\mathcal{R}_c > 1$.*

Note: case (iv) shows the possibility of the occurrence of subcritical or backward bifurcation. By backward bifurcation we mean the coexistence of a stable disease free equilibrium with a stable endemic equilibrium when $\mathcal{R}_c$. When bifurcation occurs, $\mathcal{R}_c < 1$ is only necessary but not sufficient condition for the control of diphtheria. So we need to show that $\mathcal{R}_c$ is a sufficient and necessary condition for the control of diphtheria under consideration.

Bifurcation Analysis

The diphtheria dynamics model (1) exhibit backward (Subcritical) bifurcation near $\mathcal{R}_c = 1$, that is coexistence of disease-free equilibrium and endemic equilibrium when $\mathcal{R}_c < 1$. The epidemiological consequences of backward bifurcation is that, $\mathcal{R}_c < 1$ will not guaranty the condition for the disease control. Centre manifold theorem stated by (Castillo-Charez and Son, 2002) is applied in the model (1) for bifurcation analysis, to analyse the stability near disease-free equilibrium at $\mathcal{R}_c = 1$.
let $\beta = \beta^{**}$ be the bifurcation parameter and $\mathcal{R}_c = 1$ Implies;

$$\beta^{**} = \frac{(\phi+\mu)(\tau+\gamma_1+\mu+\delta_1)}{\sigma[(1-p)(\mu+\theta\phi)+\theta p(\phi+\mu)]}. \tag{39}$$

The Theorem is applied by making the change of variables,
let,
$S_u = x_1$, $S_a = x_2$, $E = x_3$, $I = x_4$, $J = x_5$ and $R = x_6$ such that, $N = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$.
Therefore; the equation of the model (1) can be written in the form:

$$\frac{dX}{dt} = (f_1, f_2, f_3, f_4, f_5, f_6)^T, \tag{40}$$

such that

$$
\begin{aligned}
\frac{dX_1}{dt} &= f_1 = (1-p)\pi - (\phi+\mu)x_1 - \frac{\beta^{**}x_1x_4}{N},\\
\frac{dX_2}{dt} &= f_2 = \pi p + \phi x_1 - \mu x_2 - \frac{\alpha\beta^{**}x_2\theta x_4}{N},\\
\frac{dX_3}{dt} &= f_3 = \frac{\beta^{**}x_1x_4}{N} + \frac{\alpha\beta^{**}x_2\theta x_4}{N} - (\sigma+\mu)x_3,\\
\frac{dX_4}{dt} &= f_4 = \sigma x_3 - (\gamma_1+\tau+\mu+\delta_1)x_4,\\
\frac{dX_5}{dt} &= f_5 = \tau x_4 - (\gamma_2+\mu+\delta_2)x_5,\\
\frac{dX_6}{dt} &= f_6 = \gamma_2 x_4 + \gamma_2 x_5 - \mu x_6
\end{aligned}
\tag{41}
$$

Now, the Jacobian matrix of the system 41 at disease free equilibrium ϵ^0 is given by,

$$
J(\epsilon^0) = \begin{bmatrix}
-\mu-\phi & 0 & 0 & -\beta M & 0 & 0\\
\phi & -\mu & 0 & -\beta\, N & 0 & 0\\
0 & 0 & -\mu-\sigma & \beta\,(M+N) & 0 & 0\\
0 & 0 & \sigma & -\tau-\gamma_1-\mu-\delta_1 & 0 & 0\\
0 & 0 & 0 & \tau & -\gamma_2-\mu-\delta_2 & 0\\
0 & 0 & 0 & \gamma_1 & \gamma_2 & -\mu
\end{bmatrix}
\tag{42}
$$

where,

$$
M = \frac{\mu((1-p)}{\phi+\mu}, N = \frac{\mu((1-p)\phi + p(\phi+\mu))}{\mu(\phi+\mu)}
$$

.

The linearized system (42) with $\beta = \beta^{**}$ has a zero eigenvalues.Now, let $V = [v_1, v_2, v_3, v_4, v_5, v_6]$ and $W = [w_1, w_2, w_3, w_4, w_5, w_6]^T$ be the appropriate left and right eigenvectors linked to the system's Jacobian matrix's simple zero eigenvalues, respectively (42).
Solving for the right eigenvectors W we have,

$$
J(\epsilon^0).W = \begin{bmatrix}
-\mu-\phi & 0 & 0 & -\beta M & 0 & 0\\
\phi & -\mu & 0 & -\beta\, N & 0 & 0\\
0 & 0 & -\mu-\sigma & \beta\,(M+N) & 0 & 0\\
0 & 0 & \sigma & -\tau-\gamma_1-\mu-\delta_1 & 0 & 0\\
0 & 0 & 0 & \tau & -\gamma_2-\mu-\delta_2 & 0\\
0 & 0 & 0 & \gamma_1 & \gamma_2 & -\mu
\end{bmatrix}
\begin{bmatrix} w_1\\ w_2\\ w_3\\ w_4\\ w_5\\ w_6 \end{bmatrix} = 0,
\tag{43}
$$

we have,

$$
\begin{aligned}
&w_1 = \frac{-\beta^{**}Mw_4}{\phi+\mu} < 0, w_2 = \frac{\phi w_1 - \beta^{**}Mw_4}{\mu} < 0,\\
&w_3 = w_3 > 0, w_4 = w_4 > 0,\\
&w_5 = \frac{\tau w_4}{\gamma_2+\mu+\delta_1} > 0, w_6 = \frac{w_4(\gamma_1+\gamma_2)}{\mu} > 0,
\end{aligned}
$$

Similarly, solving for the left eigenvectors V

$$V^T.J(\epsilon^0) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}^T \begin{bmatrix} -\mu-\phi & 0 & 0 & -\beta M & 0 & 0 \\ \phi & -\mu & 0 & -\beta N & 0 & 0 \\ 0 & 0 & -\mu-\sigma & \beta(M+N) & 0 & 0 \\ 0 & 0 & \sigma & -\tau-\gamma_1-\mu-\delta_1 & 0 & 0 \\ 0 & 0 & 0 & \tau & -\gamma_2-\mu-\delta_2 & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & -\mu \end{bmatrix} = 0, \tag{44}$$

we have,

$$v_1 = v_2 = v_5 = v_6 = 0, v_3 = \frac{\sigma v_4}{\mu+\sigma} > 0, v_4 \quad = v_4 > 0.$$

Now, computing the partial derivatives of the system (41) which are non-zero. Since $v_1 = v_2 = 0$, and the second partial derivative of f_4, f_5 and f_6 are zeros, we only consider for $k = 3$ that is,

$$\frac{dX_3}{dt} = f_3 = \frac{\beta^{**}x_1x_4}{N} + \frac{\theta\beta^{**}x_2\theta x_4}{N} - (\sigma+\mu)x_3. \tag{45}$$

We get

$$\frac{\partial^2 f_3}{\partial x_1 \partial x_4} = \frac{\beta^{**}}{N}, \frac{\partial^2 f_3}{\partial x_2 \partial x_4} = \frac{\theta\beta^{**}}{N}, \tag{46}$$

$$\frac{\partial^2 f_3}{\partial x_4 \partial \beta} = \frac{x_1}{N}, \frac{\partial^2 f_3}{\partial x_\partial \beta} = \frac{\theta x_2}{N}, \tag{47}$$

Therefore,

$$a = v_3 \sum_{i,j=3}^{6} w_i w_j \frac{\partial^2 f_3}{\partial x_i \partial x_j}(0,0), \tag{48}$$

$$a = \frac{v_3 w_4 \beta}{N}(w_1 + \theta w_2) < 0. \tag{49}$$

Similarly

$$b = v_3 \sum_{i=3}^{6} w_i \frac{\partial^2 f_3}{\partial x_i \partial \beta^{**}}(0,0), \tag{50}$$

$$b = v_3 w_4 (\frac{S_u^0}{N^0} + \frac{\theta S_a^0}{N^0}) > 0 \tag{51}$$

As a result, $b > 0$ and $a < 0$.The following theorem is true:

Theorem 6 *The diphtheria model has a backward bifurcation at $\mathcal{R}_c = 1$. The equilibrium becomes unstable when $\beta^{**} > 0$ shifts from $\beta^{**} < 0$. A negative unstable equilibrium asymptotically transforms into a positive one. Since the bifurcation that occurred is stable, $\mathcal{R}_c < 1$ is a necessary and sufficient condition for the control of diphtheria.*

Global Asymptomatic Stability of Diphtheria Endemic Equilibrium

Theorem 7 *The diphtheria endemic equilibrium is globally asymptotically stable if the control reproduction number $\mathcal{R}_c > 1$ and unstable if $\mathcal{R}_c < 1$.*

Proof 4 *Let $\mathcal{F}$ be Goh-Volterra type of Lyapunov function as given below.*

$$\begin{aligned} \mathcal{F} = &\left(S_u - S_u^{**} - S_u^{**} ln \frac{S_u}{S_{\ u}}^{**}\right) + \left(S_a - S_a^{**} - S_a^{**} ln \frac{S_a}{S_{\ a}}^{**}\right) + \left(E - E^{**} - E^{**} ln \frac{E}{E}^{**}\right) \\ &+ \frac{(\sigma+\mu)}{\sigma}\left(I - I^{**} - I^{**} ln \frac{I}{I}*^*\right) + \frac{(\sigma+\mu)(\tau+\mu)}{\tau\sigma}\left(J - J^{**} - J^{**} ln \frac{J}{J}^{**}\right) \\ &+ \frac{(\sigma+\mu)(\tau+\mu)(\gamma_2+\mu)}{\tau\gamma_2\sigma}\left(R - R^{**} - R^{**} ln \frac{R}{R^{**}}\right). \end{aligned} \tag{52}$$

Differentiating (52) with respect to time yields

$$\begin{aligned} \dot{\mathcal{F}} = &\left(1 - \frac{S_u^{**}}{S_u}\right)\dot{S_u} + \left(1 - \frac{S_a^{**}}{S_a}\right)\dot{S_a} + \left(1 - \frac{E^{**}}{E}\right)\dot{E} + \frac{(\sigma+\mu)}{\sigma}\left(1 - \frac{I^{**}}{I}\right)\dot{I} \\ &\frac{(\sigma+\mu)(\tau+\mu)}{\tau\sigma}\left(1 - \frac{J^{**}}{J}\right)\dot{J} + \frac{(\sigma+\mu)(\tau+\mu)(\gamma_2+\mu)}{\tau\gamma_2\sigma}\left(1 - \frac{R^{**}}{R}\right)\dot{R} \end{aligned} \tag{53}$$

with

$$N = \frac{\pi}{\mu} \tag{54}$$

As the infection's force is altered, we have

$$\bar{\lambda} = \bar{\beta} I \tag{55}$$

where

$$\bar{\beta} = \beta \frac{\pi}{\mu} \tag{56}$$

When (1) is substituted with (53), we obtain

$$\begin{aligned}
\dot{\mathcal{F}} &= \left(1 - \frac{S_u^{**}}{S_u}\right)((1-p)\pi - \lambda S_u - \mu S_u) + \left(1 - \frac{S_a^{**}}{S_a}\right)(p\pi - \alpha\lambda S_a - \mu S_a) \\
&+ \left(1 - \frac{E^{**}}{E}\right)(\lambda S_u + \alpha\lambda S_a - (\sigma+\mu)E) + \frac{(\sigma+\mu)}{\sigma}\left(1 - \frac{I^{**}}{I}\right)(\sigma E - (\tau+\mu)I) \\
&+ \frac{(\sigma+\mu)(\tau+\mu)}{\sigma\tau}\left(1 - \frac{J^{**}}{J}\right)(\tau I - (\gamma_2+\mu)J) \\
&+ \frac{(\sigma+\mu)(\tau+\mu)(\gamma_2+\mu)}{\sigma\gamma_2\tau}\left(1 - \frac{R^{**}}{R}\right)(\gamma_2 J - \mu R)
\end{aligned} \tag{57}$$

With relationships

$$\begin{aligned}
(1-p)\pi = \lambda^{**}S_u^{**} + \mu S_u^{**}, p\pi = \theta\lambda^{**}S_a^{**} + \mu S_a^{**}, (\sigma+\mu)E^{**} = \lambda^{**}S_u^{**} + \theta\lambda^{**}S_a^{**}, \\
(\tau+\mu)I^{**} = \sigma E^{**}, (\gamma_2+\mu)J^{**} = \tau I^{**}, \mu R^{**} = \gamma_2 J^{**}.
\end{aligned} \tag{58}$$

The relations in (58) can be changed to (57).

$$\begin{aligned}
&\dot{\mathcal{F}} \le \mu S_u^{**}\left(2 - \frac{S_u}{S_u^{**}} - \frac{S_u^{**}}{S_u}\right) + \mu S_a^{**}\left(2 - \frac{S_a}{S_a^{**}} - \frac{S_a^{**}}{S_a}\right) \\
&+\lambda S_u^{**}\left(6 - \frac{S_u^{**}}{S_u} - \frac{S_u E^{**}}{S_u^{**}E} - \frac{EI^{**}}{E^{**}I} - \frac{IJ^{**}}{I^{**}J} - \frac{JR^{**}}{J^{**}R} - \frac{R}{R^{**}}\right) \\
&+ \theta\lambda S_a^{**}\left(6 - \frac{S_a^{**}}{S_a} - \frac{S_a E^{**}}{S_a^{**}E} - \frac{EI^{**}}{E^{**}I} - \frac{IJ^{**}}{I^{**}J} - \frac{JR^{**}}{J^{**}R} - \frac{R}{R^{**}}\right)
\end{aligned} \tag{59}$$

Furthermore, we utilize the relationship between the geometric and arithmetic means to derive

$$\begin{aligned}
&\left(2 - \frac{S_u}{S_u^{**}} - \frac{S_u^{**}}{S_u}\right) \le 0, \left(2 - \frac{S_a}{S_a^{**}} - \frac{S_a^{**}}{S_a}\right) \le 0, \\
&\left(6 - \frac{S_u^{**}}{S_u} - \frac{S_u E^{**}}{S_u^{**}E} - \frac{EI^{**}}{E^{**}I} - \frac{IJ^{**}}{I^{**}J} - \frac{JR^{**}}{J^{**}R} - \frac{R}{R^{**}}\right) \le 0, \\
&\left(6 - \frac{S_a^{**}}{S_a} - \frac{S_a E^{**}}{S_a^{**}E} - \frac{EI^{**}}{E^{**}I} - \frac{IJ^{**}}{I^{**}J} - \frac{JR^{**}}{J^{**}R} - \frac{R}{R^{**}}\right) \le 0.
\end{aligned} \tag{60}$$

Hence, we have $\dot{F} \le 0$ with conditions that $\phi = \tau = \delta_1 = \delta_2 = 0$ and $\mathcal{R}_c > 1$, since all the concerned variable in the model such as S_u, S_a, E, I, J and R are at steady state (Diphtheria endemic steady state), this can be used in place of the relevant variable of (1) to give

$$\lim_{t\to\infty} (S_u(t), S_a(t), E(t), I(t), J(t), R(t)) \to (S_u, S_a, E, I, J, R \tag{61}$$

Therefore, the result follows by applying Lasalle invariance principle (Lasalle, 1976)

Sensitivity analysis

In this section, we utilized the forward sensitivity index method to analyze the proposed diphtheria model in relation to the reproduction number $\mathcal{R}_c$ with respect to the biological parameters used in the model. The sign of each parameter was determined using this method, with negative values indicating that the parameter is most sensitive for decreasing $\mathcal{R}_c$, while positive values indicate that the parameter is most sensitive for increasing $\mathcal{R}_c$ (Andrawus *et al.*, 2024).

The normalized local sensitivity index of the $\mathcal{R}_c$ with respect to the parameters is given by,

$$\alpha_{\xi}^{\mathcal{R}_c} = \frac{\xi}{\mathcal{R}_c} \times \frac{\partial \mathcal{R}_c}{\partial \xi} \tag{62}$$

NUMERICAL SIMULATIONS

The numerical simulation of the model's state variables using the parameter values listed in Table 3 is shown in this section. The transmission dynamics of the model (1) are thoroughly understood through a numerical

Table 2: Forward Normalized Sensitivity Indices

Parameter	Elasticity Indices	Values of the Elasticity index
θ	$\alpha_{\theta}^{\mathcal{R}_c}$	0.0052
ϕ	$\alpha_{\phi}^{\mathcal{R}_c}$	-0.04018
σ	$\alpha_{\sigma}^{R_c}$	0.3112
β	$\alpha_{\beta}^{R_c}$	1.0000
τ	$\alpha_{\tau}^{R_c}$	-0.6125
γ_1	$\alpha_{\gamma_1}^{R_c}$	-0.5200
γ_2	$\alpha_{\gamma_2}^{R_c}$	-0.2652
δ_1	$\alpha_{\delta_1}^{R_c}$	0.060
δ_2	$\alpha_{\delta_2}^{R_c}$	0.0753

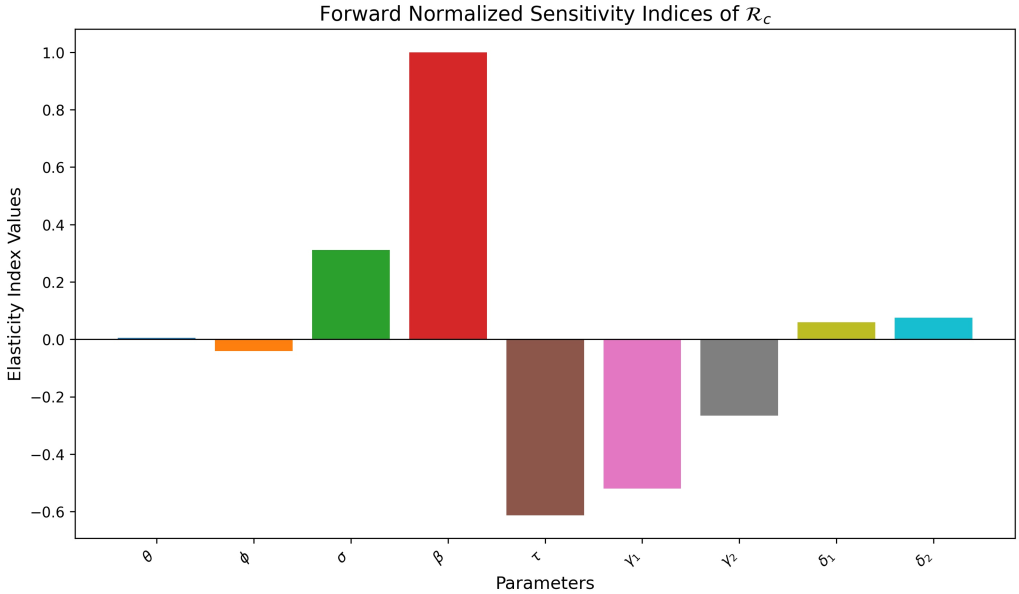

Figure 2: Bar chart graph showing the elasticity indices versus parameters

Table 3: Ranges and baseline values of parameters of model (1).

Parameter	Ranges (Baseline)	Unit	Reference
λ	0.0006	per year	Fitted
π	0.01865	per year	(Andrawus *et al,. 2025)*
p	0.06	per year	Fitted
ϕ	0.1007	per year	Fitted
μ	0.01865	per year	(Andrawus *et al,. 2025)*
δ_1	0.0713	per year	Fitted
δ_2	0.0653	per year	(Andrawus *et al,. 2025)*
β	0.221	per year	(Andrawus *et al,. 2025)*
θ	0.76521	per year	Fitted
γ_1	0.032676	per year	Fitted
γ_2	0.0746	per year	Fitted
τ	0.2	per year	Fitted
σ	0.5	per year	Fitted

simulation. Time-series diagrams are used to show how the compartments behave and how important variables affect the state variables.

DISCUSSION

The transmission dynamics of the model was simulated using state variables and the parameters in table ??. The behaviour of the state variables and pattern of movement from one compartment to another are examined. Figure 3 shows the pattern of unaware and aware susceptible individuals with different level of awareness campaign. The plots clearly shows the impact of awareness parameter ϕ. As ϕ increases, number of unaware individuals decreases while aware

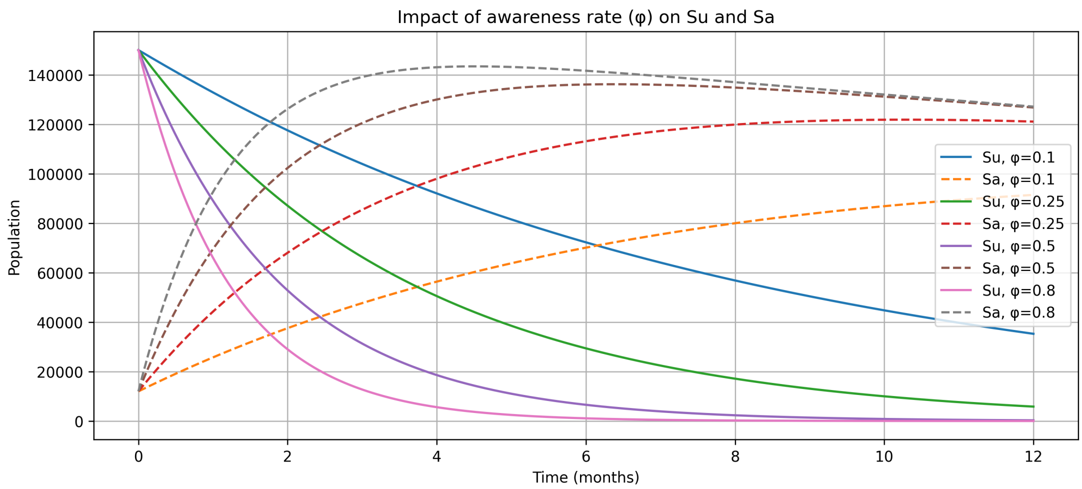

Figure 3: Pattern of Susceptible aware and unaware individuals with different values of ϕ

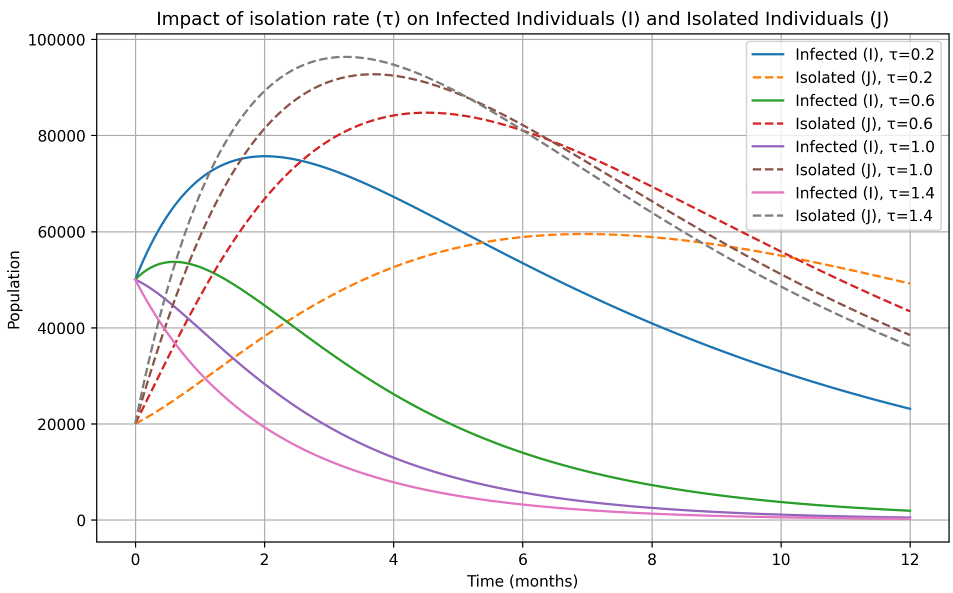

Figure 4: Pattern of infected and isolated individuals with different values of τ

individuals increases. This pattern highlights the impact of awareness campaigns on the spread of diphtheria. Figure 4 illustrates the impact of isolation parameter τ on the trends of infected and isolated individuals in the population. The plots demonstrate that as τ increases, the number of infected individuals decreases, while the number of isolated individuals increases. This trend highlights the effectiveness of isolation strategies in reducing the number of infected individuals and increasing the number of isolated individuals, which can potentially slow down the spread of diphtheria in the population.

CONCLUSION

In this paper, we developed a nonlinear deterministic model which incorporates public awareness and isolation for the transmission dynamics of diphtheria. The analysis of the model reveals that the diphtheria free equilibrium is both locally and globally asymptotically stable whenever the associated reproduction number $\mathcal{R}_c < 1$ and unstable when $\mathcal{R}_c > 1$. Contrarily, the endemic equilibrium is globally asymptotically stable when the associated reproduction number is $\mathcal{R}_c > 1$ and unstable when $\mathcal{R}_c < 1$. Furthermore, the model undergoes the phenomenon of backward bifurcation in which a stable disease-free equilibrium coexists with a stable endemic equilibrium. The epidemiological implication of backward bifurcation is $\mathcal{R}_c < 1$ is necessary but not sufficient condition for diphtheria control even when the classical requirement are satisfied, however the backward bifurcation analysis shows that when the bifurcation parameter $\beta^{**} < 0$ the system is locally asymptotically stable and there exist a positive unstable equilibrium, while If $\beta^{**} > 0$ is unstable and there exist a negative and locally asymptotically stable equilibrium. Hence the requirement of having $\mathcal{R}_c < 1$ will not suffices the condition for the control of diphtheria. The most sensitive parameters for the control of the spread of diphtheria are identified using the forward sensitivity index method as shown in Figure 2, the most sensitive parameters that increasing the transmission are β and σ, while the parameters for decreasing are τ and γ_1 respectively. In addition, the numerical simulations carried out in figure 3 show the impact of public awareness. Similarly, figure 4 show the impact of isolation. Finally, the result shows public awareness will help in curtailing the spread of diphtheria infection, and when isolation is applied to infected individuals.

References

Abubakar, A., Ahmad, A., Musa, S., Ahmad, Y.U., & Abubakar, U. (2025). Assessing the impact of booster vaccine and isolation on the transmission dynamic of pertussis.*Fudma journal of sciences*, 9(8), 392-405

Andrawus, J., Omotoso, K. I., Andrew, A. A., Eguda, F. Y., Babuba, S., & Ibrahim, K. G. (2025). Mathematical model analysis on the significance of surveillance and awareness on the transmission dynamics of diphtheria. *Journal of the Nigerian Society of Physical Sciences*, 2618-2618.

Andrawus, J., Abubakar, A., Yusuf, A., Andrew, A. A., Uzun, B., & Salahshour, S. (2024). Impact of public awareness on haemo-lyphatic and meningo-encepphalitic stage of sleeping sickness using mathematical model approach. *The European Physical Journal Special Topics*, 1-19.

Castillo-Chavez, C., & Song, B. (2004). Dynamical models of tuberculosis and their applications *Math. Biosci. Eng*, 1(2), 361-404.

Egbune, N. J., Akponana, E. B., Arierhie, E. O., & Okedoye, A. M. (2024). Mathematical analysis of spread and control of diphtheria with emphasis on diphtheria antitoxin efficiency. *European Journal of Theoretical and Applied Sciences*, 2(3), 152-172.

Elsinga, J., van Meijeren, D., & Reubsaet, F. (2023). Surveillance of diphtheria in the Netherlands between 2000–2021: cutaneous diphtheria supersedes the respiratory form. *MC Infectious Diseases*, 23(1), 1-9.

Ibrahim, K. G., Andrawus, J., Abubakar, A., Yusuf, A., Maiwa, S. I., Bitrus, K., ... & Jonathan, J. (2025). Mathematical analysis of chickenpox population dynamics unveiling the impact of booster in enhancing recovery of infected individuals.*Modeling Earth Systems and Environment*, 11(1), 46.

Izzati, N., Yannuansa, N., Ummah, I., Wati, D. A. R., Indahwati, E., Purwasih, S. M., & Kholis, N. (2024). Dynamical analysis of a mathematical model on the spread of diphtheria disease with vaccination completeness factors. *Journal Focus Action of Research Mathematic (Factor M)*, 7(2), 206-223.

Johnson, O. S., Edogbanya, H. O., Wakili, A., & John, A. T. (2024). Diphtheria disease transmission dynamics in low vaccine coverage setting. *International Journal of Mathematical Sciences and Optimization*: Theory and Applications, 10(2), 79-106

Lasalle, J.P. (1976). The stability of dynamical systems. *Regional conference series in applied mathematics*, *SIAM*, Philadelphia.

Musa, S., Usaini, S., Ahmed, I., Kiataramkul, C., & Tariboon, J. (2025). Dynamics of the Diphtheria Epidemic in Nigeria: Insights from the Kano State Outbreak Data. *Mathematics*, 13(7), 1189. https://doi.org/10.3390/math13071189

Pembroke, B., Will, R. C., Madhav, A., Shetty, V., & Mutreja, A. (2023). Diphtheria and the AB Toxin Group.

Sahib, I., Bettioui, B., Baroudi, M., Khajji, B., Faouzi, L., Alia, M., ... & Belam, M. (2024). Modeling and predicting the spread of diphtheria in guinea using delayed stochastic differential equations.*Commun. Math. Biol. Neurosci.*, 2024, Article-ID.

Traugott, M.T., Pleininger, S., Inschlag-Tisch, S., Eder, B., Seitz, T., Merrelaar, A., Reiß-Kornfehl, J., Fussi, J., Schindler, S., Blaschitz, M. and Heger, F., (2023). A case of fulminant respiratory diphtheria in a 24-year-old Afghan refugee in Austria in May 2022: a case report. *Infection*, 51(2), pp.489-495.

World Health Organization, (2023). Vaccine-preventable disease update: reported diphtheria cases in the WHO European Region, 2022: 19 January 2022 (No. WHO/EURO: 2023-6208-45973-68002). World Health Organization. Regional Office for Europe.

World Health Organization (2024).